π and sums of squares

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An ancient textbook

A palm-leaf manuscript called the **Yuktibhasha** written around 1530 by Jyeshthadeva, in the language Malayalam from the state of Kerala, India:

Madhava's verse

A verse attributed to **Madhava** of Sangamagrama who lived around 1340-1425:

[1]വ്യാസാർദ്ധേന ഹതാദഭീഷ്ടഗ്ഗുണതഃ
കോട്യാപ്തമാദ്യം ഫലം
ജ്യാവർഗ്ഗേണ വിനിഘ്നമാദിമഫലം
തത്തത്ഫലം ചാഹരേത് ।

കൃത്വാ കോടിഗുണസ്യ തത്ര തു ഫലേ-
ഷ്വേകത്രിപഞ്ചാദിഭി-
ർഭക്തേഷ്വോജയുതൈസ്ത്യജേത് സമയുതിം
ജീവാധനുശ്ശിഷ്യതേ ॥

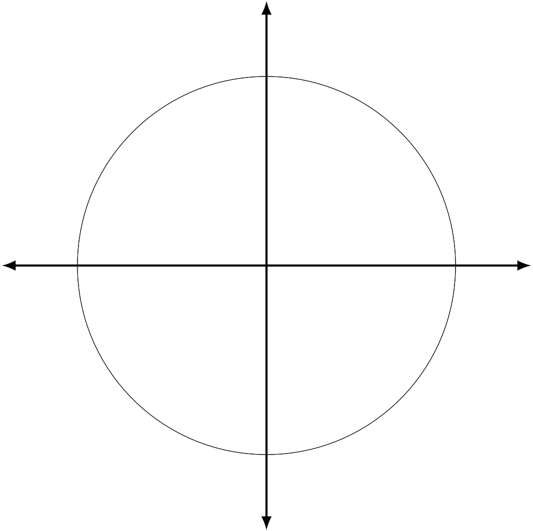
Take any circular arc, whose abscissa is not less than its ordinate. Multiply the ordinate of the arc by the semidiameter and divide it by the abscissa. This gives the first term. Multiply this term by the square of the ordinate and divide it by the square of the abscissa; a second term results. Repeat the process of multiplying by the square of the ordinate and dividing by the square of the abscissa. Thus obtain successive terms and divide them in order by the odd integers $1, 3, 5, \ldots$. *If now the terms whose order is odd are added to, and the terms whose order is even subtracted from the preceding, what remains is the length of the circular arc.*

We get

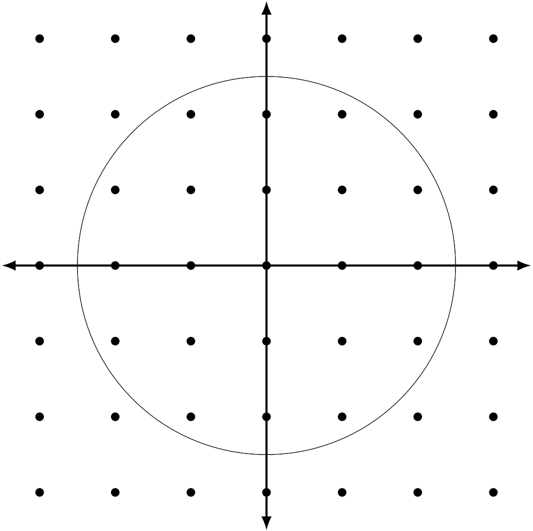
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

We will try to obtain this series using some number theory

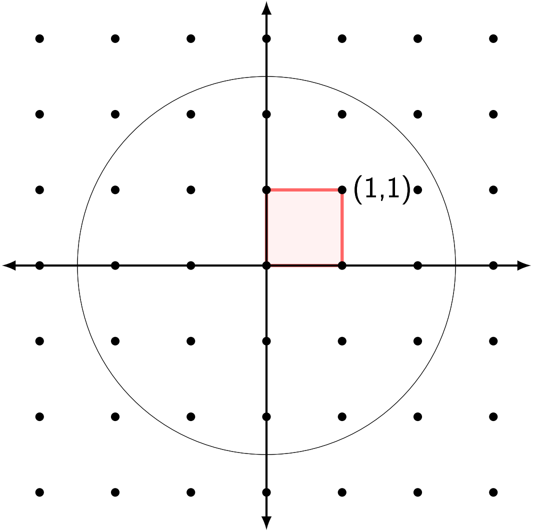
Counting points inside a circle



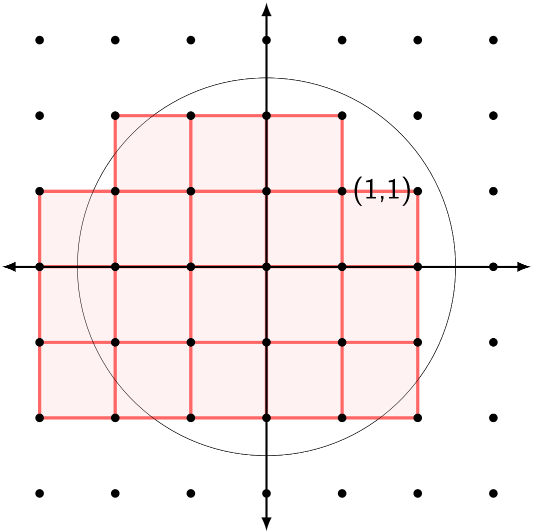
Counting points inside a circle



Counting points inside a circle



Counting points inside a circle



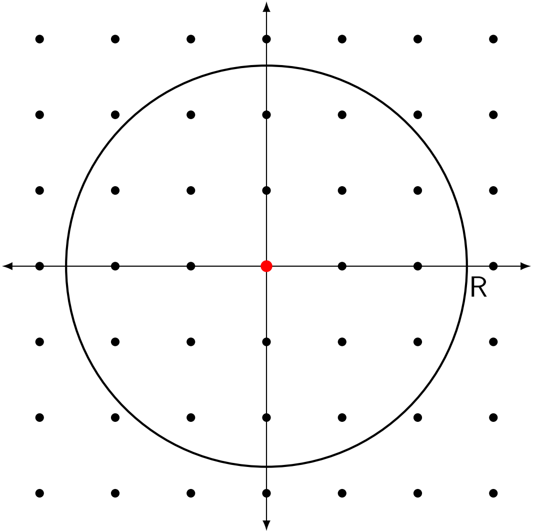
Expression for π

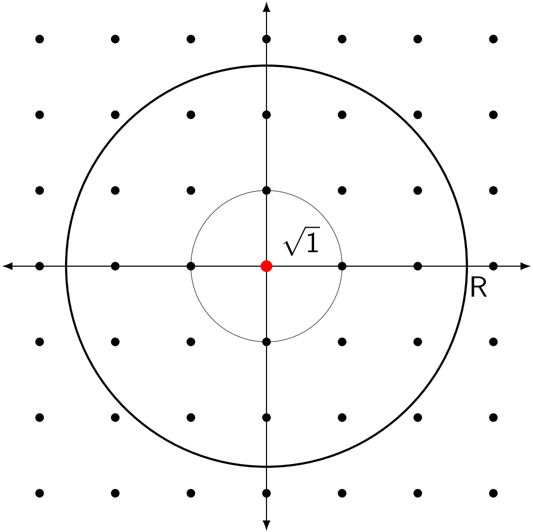
$$\pi R^2 = \{\# \text{ of lattice points inside}\} + \text{Error}$$

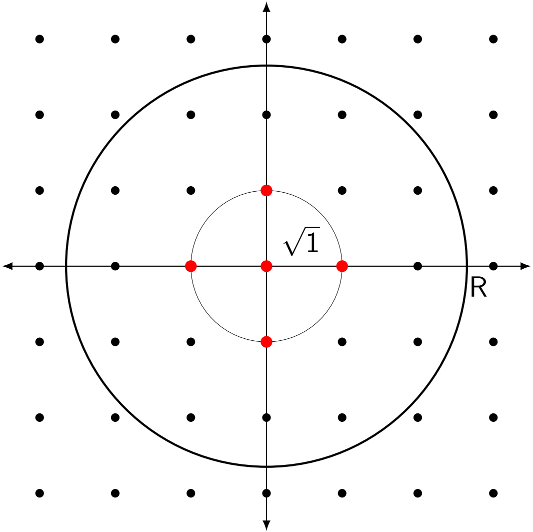
$$\pi = \lim_{R\to\infty} \frac{\{\# \text{ of lattice points inside}\}}{R^2},$$

provided we have

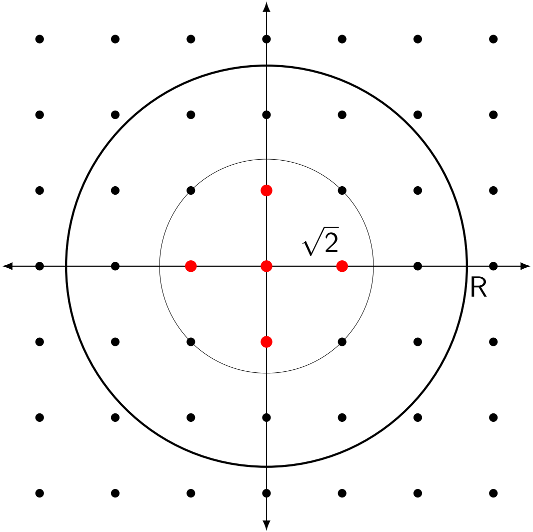
$$\lim_{R\to\infty} \frac{\text{Error}}{R^2} = 0.$$


R

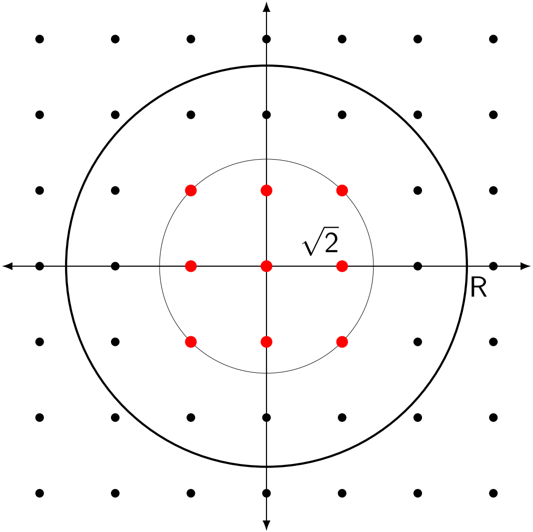

$\sqrt{1}$
R



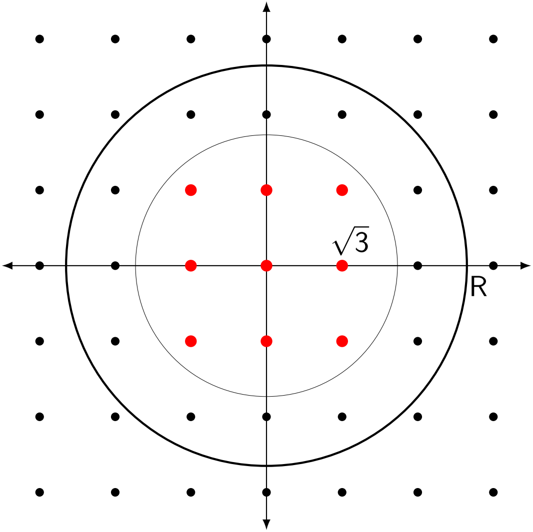
$\sqrt{1}$
R



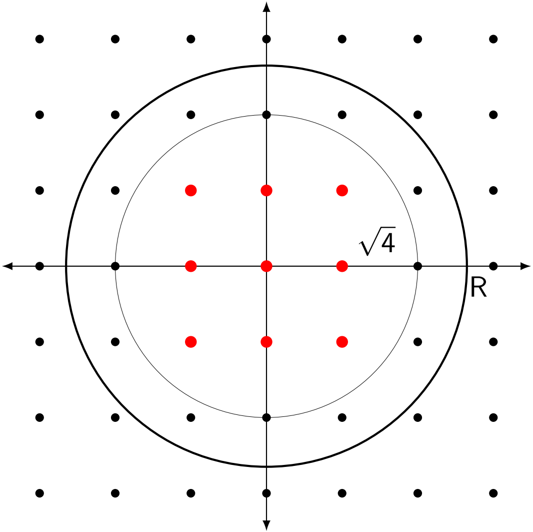
$\sqrt{2}$
R

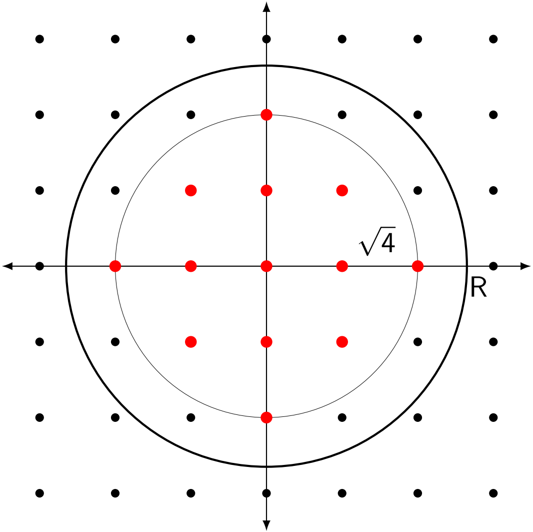


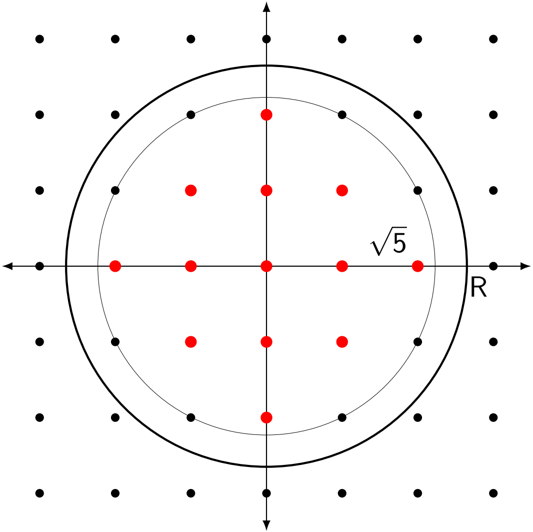
$\sqrt{2}$
R



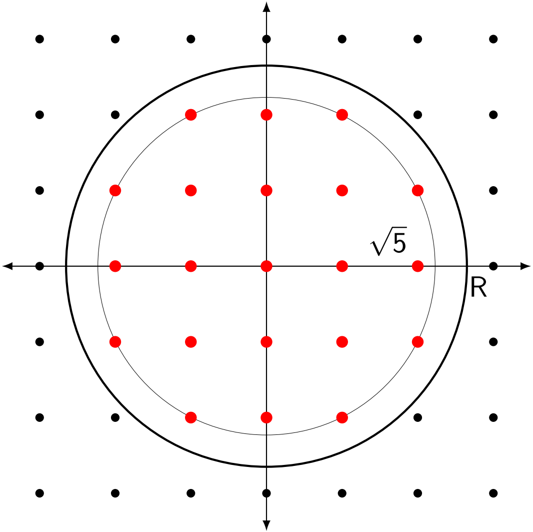
√3
R


$\sqrt{4}$
R

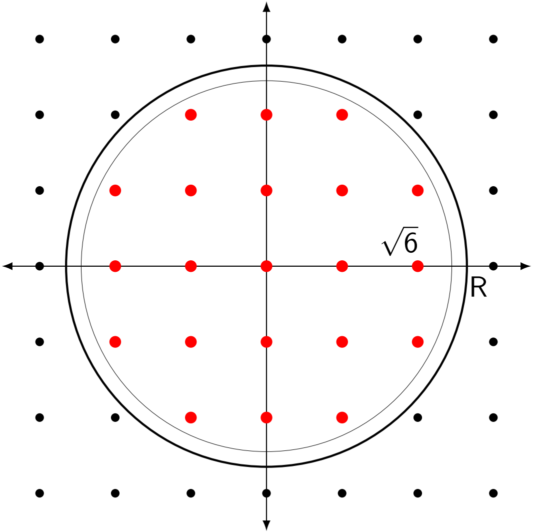

$\sqrt{4}$
R

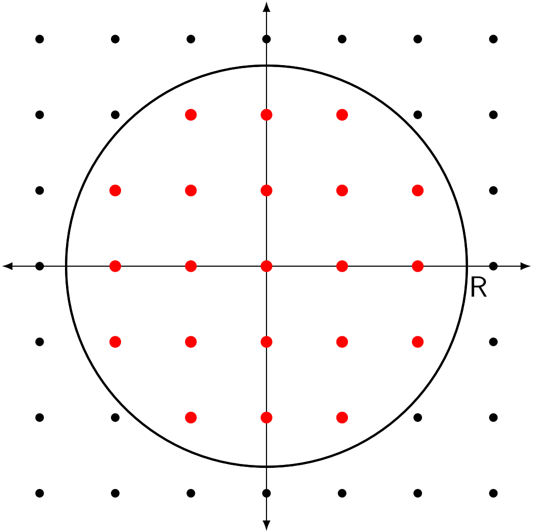


$\sqrt{5}$
R



$\sqrt{5}$
R


$\sqrt{6}$
R


R

Sum of squares

Boils down to counting how many ways to write

$$n = a^2 + b^2,$$

with $a, b \in \mathbb{Z}$, and $\sqrt{n} \leq R$.
Let's write

$$n = 2^m \cdot p_1^{e_1} p_2^{e_2} \cdots p_j^{e_j} \cdot q_1^{f_1} q_2^{f_2} \cdots q_k^{f_k},$$

where $p_i = 4c + 1$ and $q_i = 4d + 3$.

Theorem (Fermat's Christmas Theorem)

An integer n can be written as the sum of two square integers as $n = a^2 + b^2$ if and only if the exponents f_i are all even.

For example $3 = a^2 + b^2$ has no solutions. Same for $6 = 2 \cdot 3 = a^2 + b^2$.
But

$$90 = 2 \cdot 3^2 \cdot 5 = 9^2 + 3^2.$$

Counting the number of ways

It remains to find a nice method to count the number of ways to write $n = a^2 + b^2$. We look at points in the first quadrant.

$$
\begin{aligned}
5 &\to (1,2);\ (2,1) \\
5^2 &\to (0,5);\ (3,4);\ (4,3) \\
5^3 &\to (2,11);\ (11,2);\ (5,10);\ (10,5) \\
5^k &\to \{k+1 \text{ points}\}
\end{aligned}
$$

$$
\begin{aligned}
7 &\to ☹ \\
7^2 &\to (0,7^2) \\
7^3 &\to ☹ \\
7^k &\to \begin{cases} ☹ & \text{if } k \text{ is odd,} \\ (0,7^k) & \text{if } k \text{ is even.} \end{cases}
\end{aligned}
$$

For 2^k we always have exactly one point, for all k.

Capturing all this information

Define, for a prime p

$$\chi(p) = \begin{cases} 1 & \text{if } p = 4m+1, \\ -1 & \text{if } p = 4m+3, \\ 0 & \text{if } p \text{ is even.} \end{cases}$$

We let $\chi(p^k) = \chi(p)^k$ and $\chi(1) = 1$.
Then, for an integer n, the number of ways $r(n)$ to write it as a sum of two squares is

$$r(n) = 4 \sum_{d|n} \chi(d).$$

For each n, such that $1 \leq \sqrt{n} \leq R$, we count $r(n)$ points.

Putting it all together

$$\pi R^2 \approx \sum_{\sqrt{n}\le R} r(n) = 4\left(\chi(1)\cdot R^2 + \chi(2)\frac{R^2}{2} + \chi(3)\frac{R^2}{3} + \chi(4)\frac{R^2}{4} + \cdots\right)$$
$$= 4R^2\left(\chi(1) + \frac{\chi(2)}{2} + \frac{\chi(3)}{3} + \frac{\chi(4)}{4} + \cdots\right)$$
$$= 4R^2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right).$$

and thus,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

provided the error term satisfies

$$\lim_{R\to\infty} \frac{\text{Error}}{R^2} = 0.$$

We can define an L-**function** as

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

1. Encodes important information about the set $\mathbb{Z}[i]$. Dirichlet class number formula
2. The Riemann zeta function is an L-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Carries information about the prime numbers. Prime number theorem.

C'EST TOUT!
MERCI!