The Mahler measure of some polynomial families

Siva Sankar Nair

(including joint work with Matilde Lalín and Subham Roy)

Université de Montréal

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The definition

For a non-zero rational function $P \in \mathbb{C}(x_1, \ldots, x_n)^\times$, we define the (logarithmic) **Mahler measure** of P to be

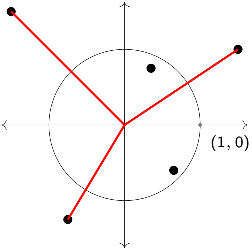
$$\mathfrak{m}(P) := \int_{[0,1]^n} \log \left| P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n}) \right| \, d\theta_1 \cdots d\theta_n.$$

- Average value of $\log |P|$ over the unit n-torus.
- Introduced as a height function

The one-variable case

If $P(x) = A\prod_{j=1}^{d}(x-\alpha_j)$, then Jensen's formula implies

$$\mathfrak{m}(P) = \int_0^1 \log|P(e^{2\pi i\theta})|d\theta = \log|A| + \sum_{\substack{j \\ |\alpha_j|>1}} \log|\alpha_j|.$$



- Thus, if $P(x) \in \mathbb{Z}[x] \Longrightarrow \mathfrak{m}(P) \geq 0$

Some Properties

- Kronecker's Lemma: $P \in \mathbb{Z}[x]$, $P \neq 0$,

$$\mathfrak{m}(P) = 0 \text{ if and only if } P(x) = x^n \prod_i \Phi_i(x),$$

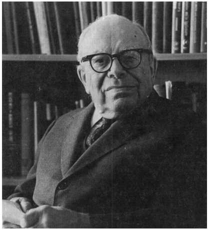
where $\Phi_i(x)$ are cyclotomic polynomials.

- Lehmer's Question (1933, still open):
Does $\exists$ a $\delta > 0$ such that, for any $P \in \mathbb{Z}[x]$,
if $\mathfrak{m}(P) \neq 0$, then $\mathfrak{m}(P) > \delta$?

$$\mathfrak{m}(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1) \approx 0.162357612\ldots$$

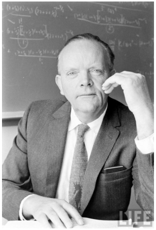
- Related to heights. For an algebraic integer α with logarithmic Weil height $h(\alpha)$,

$$\mathfrak{m}(f_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]h(\alpha).$$



Kurt Mahler

Johan Jensen



Derrick Lehmer

More variables, more problems (more fun?)

Calculating the Mahler measure of multi-variable polynomials is very difficult.
For certain polynomials, the Mahler measure comes up as a value of an L-function!
Smyth, 1981:

- $$\mathfrak{m}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

- $$\mathfrak{m}(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3) = -14\zeta'(-2)$$

More examples

Condon, 2004:

- $$\mathfrak{m}(x+1+(x-1)(y+z)) = \frac{28}{5\pi^2}\zeta(3) = -\frac{112}{5}\zeta'(-2)$$

Lalín, 2006:

- $$\mathfrak{m}\left(1+x+\left(\frac{1-v}{1+v}\right)\left(\frac{1-w}{1+w}\right)(1+y)z\right) = \frac{93}{\pi^4}\zeta(5) = 124\zeta'(-4)$$

Rogers and Zudilin, 2010:

- $$\mathfrak{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+8\right) = \frac{24}{\pi^2}L(E_{24a3},2) = 4L'(E_{24a3},0)$$

Matilde Lalín

Chris Smyth

David Boyd

Coming up with such identities

- In general, Mahler measures are arbitrary real values.
- Polynomials with a certain structure may give interesting values.
- Use the computer to compare with known L-values.
- Commonly associated to evaluating certain *polylogarithms*.

An explanation for the appearance of L-values

Let $P = A_d y_{n+1}^d + A_{d-1} y_{n+1}^{d-1} + \cdots + A_0 \in \mathbb{C}[y_1, \ldots, y_{n+1}]$ and

$$D = \{(y_1, \ldots, y_n, y_{n+1}) : \forall i \leq n, |y_i| = 1,\ |y_{n+1}| > 1, P(y_1, \ldots, y_{n+1}) = 0\}$$

Theorem (Deninger 1997)

If P is irreducible, then

$$\mathfrak{m}(P) = \mathfrak{m}(A_d) + \frac{(-1)^n}{(2\pi i)^n} \int_{\overline{D}} \eta(y_1, \ldots, y_{n+1}).$$

Here $\eta(y_1, \ldots, y_{n+1})$ is a closed differential form that satisfies

$$\eta(y_1, \ldots, y_{n+1})|_D = (-1)^n \log |y_{n+1}| \frac{\mathrm{d}y_1}{y_1} \wedge \cdots \wedge \frac{\mathrm{d}y_n}{y_n}.$$

Can be related to a Beilinson regulator. $\longrightarrow$ Beilinson conjectures

Calculations by Brunault and Zudilin

Numerical calculations by Brunault and Zudilin:

$$\left.\begin{array}{l} \mathfrak{m}(x^2+x+1+(x^2-1)(y+z)) \\ \mathfrak{m}(x^3-x^2+x-1+(x^3+1)(y+z)) \\ \mathfrak{m}(x^4-x^3+x-1+(x^4-x^2+1)(y+z)) \\ \mathfrak{m}(x^4-x^3+x-1+(x^4-x^3+x^2-x+1)(y+z)) \\ \mathfrak{m}(x^4-x^3+x^2-x+1+(x^4-1)(y+z)) \\ \mathfrak{m}(x^4-x^3+x-1+(x^4+1)(y+z)) \\ \mathfrak{m}(x^5-x^4+x-1+(x^5+1)(y+z)) \end{array}\right\} \stackrel{?}{=} \frac{28}{5\pi^2}\zeta(3).$$

Condon showed

$$\mathfrak{m}(x+1+(x-1)(y+z)) = \frac{28}{5\pi^2}\zeta(3).$$

Francois Brunault

Wadim Zudilin

Is there some connection?

$$x+1+(x-1)(y+z) \xrightarrow{x=\frac{X(2X+1)}{X+2}} 2\frac{X^2+X+1+(X^2-1)(y+z)}{X+2}$$

$$x+1+(x-1)(y+z) \xrightarrow{x=\frac{X(2X^2-X+1)}{-(X^2-X+2)}} 2\frac{X^3-X^2+X-1+(X^3+1)(y+z)}{-(X^2-X+2)}$$

$$x+1+(x-1)(y+z) \xrightarrow{x=\frac{X(2X^3-X^2-X+1)}{-(X^3-X^2-X+2)}} 2\frac{X^4-X^3+X-1+(X^4-X^2+1)(y+z)}{-(X^3-X^2-X+2)}$$

$$x = \frac{f(X)}{g(X)}$$

$f(X)$ → reverse the coefficients of g and multiply by a power of X

$g(X)$ → has all roots outside the unit disc

An invariant property

Theorem (Lalín & N., 2023)

Let $P(x, y_1, \ldots, y_n)$ be a polynomial over $\mathbb{C}$ in the variables $x, y_1, \ldots, y_n$. Let $g(x) \in \mathbb{C}[x]$ be such that all the roots have absolute value greater than or equal to one, let k be an integer such that $k > \deg(g)$ and let $f(x) = \lambda x^k \overline{g}(x^{-1})$, where λ is a complex number with absolute value one. We denote by $\widetilde{P}$ the rational function obtained by replacing x by $f(x)/g(x)$ in P. Then

$$\mathfrak{m}(P) = \mathfrak{m}(\widetilde{P}).$$

Families of polynomials with arbitrarily many variables

Let

$$P_k = y + \left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_k}{1+x_k}\right).$$

Theorem (Lalín, 2006)

$$\mathfrak{m}(P_{2n}) = \sum_{h=1}^{n} \frac{a_{n,h}}{\pi^{2h}}\zeta(2h+1),$$

and

$$\mathfrak{m}(P_{2n+1}) = \sum_{h=0}^{n} \frac{b_{n,h}}{\pi^{2h+1}}L(\chi_{-4}, 2h+2).$$

$a_{j,k}, b_{j,k} \in \mathbb{Q}$ related to coefficients of elementary symmetric polynomials.

Proof

$$P_k = y + \left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_k}{1+x_k}\right).$$

compare with

$$Q_\gamma(y) = y + \gamma$$

$$\mathfrak{m}(P_k) = \frac{1}{(2\pi)^k}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\mathfrak{m}\left(Q_{\left(\frac{1-e^{i\theta_1}}{1+e^{i\theta_1}}\right)\cdots\left(\frac{1-e^{i\theta_k}}{1+e^{i\theta_k}}\right)}(y)\right)\mathrm{d}\theta_1\ldots\mathrm{d}\theta_k$$

"clever" transformations

$$= \frac{2^k}{\pi^k}\int_0^\infty\cdots\int_0^\infty \mathfrak{m}(Q_{y_k})\frac{y_1\mathrm{d}y_1}{(y_1^2+1)}\cdot\frac{y_2\mathrm{d}y_2}{(y_2^2+y_1^2)}\cdots\frac{\mathrm{d}y_k}{(y_k^2+y_{k-1}^2)}.$$

We have

$$\int_0^\infty \cdots \int_0^\infty \mathfrak{m}(Q_{y_k}) \frac{y_1 \mathrm{d}y_1}{(y_1^2+1)} \cdot \frac{y_2 \mathrm{d}y_2}{(y_2^2+y_1^2)} \cdots \frac{\mathrm{d}y_k}{(y_k^2+y_{k-1}^2)}$$

which can be written as a linear combination of integrals of the form

$$\int_0^\infty \mathfrak{m}(Q_t) \log^j t \frac{\mathrm{d}t}{t^2 \pm 1},$$

and using

$$\int_0^1 \log^k t \, \frac{1}{t-a} \mathrm{d}t = (-1)^{k+1}(k!) \, \mathrm{Li}_{k+1}(1/a),$$

$\rightarrow$ gives zeta values and L-values

Extending these results

Lalín also looked at

$$S_{n,r} = (1+x)z + \left[\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)\right]^r (1+y).$$

$\big\downarrow$ compare with

$$Q_\gamma(x,y,z) = (1+x)z + \gamma(1+y)$$

Theorem (Lalín, N., Roy, 2024+)

For $n \geq 1$,

$$\mathfrak{m}(S_{2n,r}) = \sum_{h=1}^{n} \frac{a'_{n,h}}{\pi^{2h}}\,\mathcal{C}_r(h),$$

and for $n \geq 0$,

$$\mathfrak{m}(S_{2n+1,r}) = \sum_{h=0}^{n} \frac{b'_{n,h}}{\pi^{2h+1}}\,\mathcal{D}_r(h)$$

$$
\begin{aligned}
\mathcal{C}_r(h) :=& r(2h)!\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1)\\
&+\frac{r^2(2h-1)!}{\pi^2}\times\\
&\left\{\frac{(-1)^{h+1}7B_{2h}\pi^{2h}}{2r^2(2h)!}\zeta(3)\left(2^{2h-1}+(-1)^r2^{2h-1}+(-1)^{r+1}\right)\right.\\
&+(2h+2)(2h+1)\frac{1-2^{-2h-3}}{r^{2h+2}}(1-(-1)^r)\zeta(2h+3)\\
&-\sum_{\ell=0}^{2r-1}(-1)^{\ell}\left[\sum_{t=2}^{2h+2}\left(\frac{(t-1)(t-2)}{2}(-1)^t\left(\mathrm{Li}_t(\xi_{2r}^{\ell})-\mathrm{Li}_t(-\xi_{2r}^{\ell})\right)\right.\right.\\
&\left.\left.\left.-\binom{t-1}{2h-1}(2-2^{1-t})\zeta(t)\right)\times\frac{(2\pi i)^{2h+3-t}}{(2h+3-t)!}B_{2h+3-t}\left(\frac{\ell}{2r}\right)\right]\right\}.
\end{aligned}
$$

Examples

$$\mathfrak{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\right]^2(1+y)z\right)=\frac{21}{2\pi^2}\zeta(3)$$

$$\mathfrak{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\right]^2(1+y)z\right)=\frac{96}{\pi^3}L(\chi_{-4},4)-\frac{21}{2\pi^2}\zeta(3)$$

$$\mathfrak{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_3}{1+x_3}\right)\right]^2(1+y)z\right)=\frac{31}{2\pi^4}\zeta(5)-\frac{96}{\pi^3}L(\chi_{-4},4)+\frac{21}{2\pi^2}\zeta(3)$$

$$\mathfrak{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right)=\frac{24}{\pi^3}L(\chi_{-4},4)$$

$$\mathfrak{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^2(1+y)z\right)=\frac{21}{2\pi^2}\zeta(3)$$

$$\mathfrak{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^3(1+y)z\right)=-\frac{8}{\pi^3}L(\chi_{-4},4)+\frac{12\sqrt{3}}{\pi^2}L(\chi_{12}(11,\cdot),3)$$

$$\mathfrak{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^4(1+y)z\right)=-\frac{105}{2\pi^2}\zeta(3)+\frac{64\sqrt{2}}{\pi^2}L(\chi_8(5,\cdot),3)$$

Matilde Lalín

Subham Roy

N.

Making some clever transformations!

Why does this work – Möbius transformations?

The transformation

$$\phi(z) = \frac{1-z}{1+z}$$

sends the unit circle to the imaginary axis. For $z = e^{i\theta}$,

$$\frac{1-z}{1+z} = -2i\tan\left(\frac{\theta}{2}\right).$$

Some natural questions:

- Transformations that send unit circle to other lines?
- Those that preserve the unit circle?
  - These are

    $$\phi(z) = e^{i\alpha}\frac{z-a}{1-\bar{a}z},$$

    where $a \in \Delta$.

We've already seen this

Theorem (Lalín & N., 2023)

Let $P(x, y_1, \dots, y_n) \in \mathbb{C}[x, y_1, \dots, y_n]$, $g(x) \in \mathbb{C}[x]$ *without any root inside the unit circle,* k *be such that* $k > \deg(g)$ *and* $f(x) = \lambda x^k \overline{g}(x^{-1})$, *where* $|\lambda| = 1$. *We denote by* $\widetilde{P}$ *the rational function obtained by replacing* x *by* $f(x)/g(x)$ *in* P. *Then*

$$\mathfrak{m}(P) = \mathfrak{m}(\widetilde{P}).$$

$f(X)/g(X)$ has the form:

$$X^{k-\deg(g)} \lambda \prod_{\ell=1}^{d} \left(\frac{1 - X\overline{\gamma_j}}{X - \gamma_\ell} \right).$$

Other results

Let

$$Q_k(z_1,\ldots,z_k,y) = y + \left(\frac{z_1+\alpha}{z_1+1}\right)\cdots\left(\frac{z_k+\alpha}{z_k+1}\right),$$

where $\alpha = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}$.

Theorem (N., 2023+)

$$\mathfrak{m}(Q_{2n}) = \sum_{h=1}^{n} \frac{a_{n,h}}{\pi^{2h}}\zeta(2h+1) + \sum_{h=0}^{n-1} \frac{b_{n,h}}{\pi^{2h+1}} L(\chi_{-3}, 2h+2),$$

and

$$\mathfrak{m}(Q_{2n+1}) = \sum_{h=1}^{n} \frac{c_{n,h}}{\pi^{2h}}\zeta(2h+1) + \sum_{h=0}^{n} \frac{d_{n,h}}{\pi^{2h+1}} L(\chi_{-3}, 2h+2),$$

where $a_{l,k}, b_{l,k}, c_{l,k}, d_{l,k} \in \mathbb{R}$ are defined recursively.

Examples

We have the first few examples in this family:

$$\mathfrak{m}(P_1) = \frac{5\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$$

$$\mathfrak{m}(P_2) = \frac{91}{18\pi^2}\zeta(3) + \frac{5}{4\sqrt{3}\pi} L(\chi_{-3}, 2)$$

$$\mathfrak{m}(P_3) = \frac{91}{36\pi^2}\zeta(3) + \frac{5}{4\sqrt{3}\pi} L(\chi_{-3}, 2) + \frac{153\sqrt{3}}{16\pi^3} L(\chi_{-3}, 4)$$

$$\mathfrak{m}(P_4) = \frac{91}{36\pi^2}\zeta(3) + \frac{3751}{108\pi^4}\zeta(5) + \frac{35}{36\sqrt{3}\pi} L(\chi_{-3}, 2) + \frac{51\sqrt{3}}{8\pi^3} L(\chi_{-3}, 4)$$

Further questions

- Can we do this for other roots of unity? A general method?
- Do the coefficients have an elegant closed formula?
- Simplifying the polylog expressions
- Can we relate the complex polynomials to integer polynomials?
- Other transformations that can make this method work?

THANK YOU!