The Mahler Measure of Some Three-Variable Polynomials

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We may define for a polynomial $P(x) \in \mathbb{C}[x]$,

$$m(P) := \int_0^1 \log|P(e^{2\pi i\theta})|d\theta,$$

to be the *(logarithmic)* **Mahler Measure** of P.

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- **Kronecker's Lemma**

  For $P \in \mathbb{Z}[x]$, $m(P) = 0$ if and only if P is a product of a monomial and cyclotomic polynomials.

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  So far the polynomial with the smallest Mahler measure is

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) \approx 0.162357612\ldots$$

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\begin{aligned}
m(P) :&= \int_{[0,1]^n} \log \left| P(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}) \right| \, d\theta_1 \cdots d\theta_n \\
&= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \, \frac{dx_1}{x_1} \cdots \frac{dx_1}{x_n}.
\end{aligned}
$$

$\mathbb{T}^n = \{ (x_1, \ldots, x_n) : |x_i| = 1 \}$

unit-n-torus.

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$$\lim_{k_2 \to \infty} \cdots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \dots, x^{k_n})) = m(P(x_1, x_2, \dots, x_n)),$$

where the k_i's vary independently.

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- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1)$$

- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3) = -14\zeta'(-2)$$

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3

Matilde Lalín, 2006:

- $$m\left(1+x+\left(\frac{1-v}{1+v}\right)\left(\frac{1-w}{1+w}\right)(1+y)z\right) = \frac{93}{\pi^4}\zeta(5) = 124\zeta'(-4)$$

5

$\zeta(3) = \cancel{\Sigma}$ $\displaystyle\iiint_{0<x<y<z<1} \frac{dx\,dy\,dz}{(1-x)yz}$ $\zeta(3)=$

Polylogarithms

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For a positive integer k and $|z| < 1$, we define

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- These functions can be analytically continued beyond the unit disc, however, a branch cut is required.

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where B_j is the j^{th} Bernoulli number, $\mathrm{Li}_0(z) := -1/2$ and $\mathrm{Re}_k(z)$ denotes $\mathrm{Re}(z)$ or $\mathrm{Im}(z)$ when k is odd or even respectively.

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- For $k = 2$, we get the Bloch-Wigner dilogarithm

$$D(z) := P_2(z) = \mathrm{Im}(\mathrm{Li}_2(z) - \log|z|\mathrm{Li}_1(z)).$$

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$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0$$

- These functions satisfy several functional equations, for example

$$P_k\left(\frac{1}{z}\right) = P_k(\bar{z}) = (-1)^{k-1} P_k(z),$$

along with functional equations that depend on the weight k.

Two-Variable Case

One can show that if $P \in \mathbb{C}[x, y]$ is of the form

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$$P(x, y) = P^*(x)y^d + \cdots + a_0(x), = P^* \prod_{i=1}^{d} (y - \alpha_i(x))$$

alg. functions in x

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and $\eta(x, y)$ is a real closed differential 1-form on $C \setminus \{\text{poles and zeros of } x \text{ and } y\}$ given by

$$\eta(x, y) = \log|x| \mathrm{d} \arg y - \log|y| \mathrm{d} \arg x.$$

$P = x + y - 1$

$\eta(x,y) = \eta(x, 1-x) = dD(x)$

$m(1+x+y) = \frac{-1}{2\pi} D(\zeta \gamma)$

$x = e^{2\pi i\theta}$, $|y| = |1 - e^{2\pi i \theta}| \geq 1$

$\theta \in [\frac{1}{6}, \frac{5}{6}]$

ζ_6 $\bar{\zeta}_6$

$\frac{-1}{2\pi}\left(D(\zeta_6) - D(\bar{\zeta}_6)\right) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$

The reason this could be helpful is because

$$\eta(x, 1-x) = \mathrm{d}\, D(x),$$

so that in some cases, η is exact and we can use Stokes' Theorem to calculate the integral.

$$m(1+x+y) = m(x+y-1) = \bigcirc\, D(\zeta_6)$$

$$m(1+x+y+z) = \frac{1}{4\pi^2}\, 8\left(P_3(1) - P_3(-1)\right)$$

golden ratio

$$m(1+x+(x-1)(y+z)) = \frac{4}{\pi^2}\left(2P_3(1) - P_3(\phi) - P_3(-\phi)\right)$$

Numerical Computation

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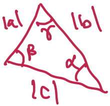
Theorem

Let $a + by + cz \in \mathbb{C}[y, z]$ with a, b, c non-zero. Then

$$\pi m(a+by+cz) = \begin{cases} \underline{D\left(\left|\frac{a}{b}\right| e^{i\gamma}\right)} + \alpha \log|a| + \beta \log|b| + \gamma \log|c| & \text{if } \Delta, \\ \pi \log \max\{\underline{|a|}, \underline{|b|}, \underline{|c|}\} & \text{if not } \Delta. \end{cases}$$

Here Δ denotes the condition that $|a|, |b|$ and $|c|$ form a triangle with angles α, β and γ.



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$$m(a(x)+b(x)y+c(x)z)=\frac{1}{\pi}\int_0^{\pi} m(a(e^{i\theta})+b(e^{i\theta})y+c(e^{i\theta})z)d\theta.$$

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In the triangular case, the integral is difficult to compute and is typically done numerically

Calculations by Brunault and Zudilin

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$$\left.\begin{array}{l} m(x^2 + x + 1 + (x^2 - 1)(y + z)) \\ m(x^3 - x^2 + x - 1 + (x^3 + 1)(y + z)) \\ m(x^4 - x^3 + x - 1 + (x^4 - x^2 + 1)(y + z)) \\ m(x^4 - x^3 + x - 1 + (x^4 - x^3 + x^2 - x + 1)(y + z)) \\ m(x^4 - x^3 + x^2 - x + 1 + (x^4 - 1)(y + z)) \\ m(x^4 - x^3 + x - 1 + (x^4 + 1)(y + z)) \\ m(x^5 - x^4 + x - 1 + (x^5 + 1)(y + z)) \end{array}\right\}$$

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Is there some connection between these polynomials?

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- The differential dx/x is transformed to

$$\frac{dx}{x} = 2\frac{dX}{X} - \frac{dX}{X+2} - \frac{dX}{X^2(X^{-1}+2)}$$

$$\oint\oint\oint \log|x+1+(x-1)(y+z)|\,\frac{dx}{x}\frac{dy}{y}\frac{dz}{z}$$

$$\oint\oint\oint \log|x+1+(x-1)(y+z)|\,\frac{dx}{x}\frac{dy}{y}\frac{dz}{z} = \frac{28}{5\pi^2}\zeta(3)$$

$$\Big\downarrow x=\frac{X(2X+1)}{X+2}$$

$$\frac{1}{2}\oint\oint\oint \log\left|\frac{2(X^2+X+1+(X^2-1)(y+z))}{X+2}\right|\left(2\frac{dX}{X}-\frac{dX}{X+2}-\frac{dX}{X^2(X^{-1}+2)}\right)\frac{dy}{y}\frac{dz}{z}.$$

$$\Big\downarrow X=1/t$$

$$|X+2| = |2t+1|$$

$$|X|=1$$

$$2\frac{dX}{X} - 2\frac{dX}{X+2}$$

Simplifying gives

$$\oint\oint\oint \log\left|\frac{2(X^2+X+1+(X^2-1)(y+z))}{X+2}\right|\left(\frac{dX}{X}-\frac{dX}{X+2}\right)\frac{dy}{y}\frac{dz}{z}.$$

0

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So, if we show that the second term evaluates to zero, then we obtain

$$\frac{28}{5\pi^2}\zeta(3)=\oint\oint\oint \log\left|\frac{2(X^2+X+1+(X^2-1)(y+z))}{X+2}\right|\frac{dX}{X}\frac{dy}{y}\frac{dz}{z}$$

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$$\oint \oint \oint \log \left| \frac{2(X^2 + X + 1 + (X^2 - 1)(y+z))}{X+2} \right| \left(\frac{dX}{X+2} \right) \frac{dy}{y} \frac{dz}{z} = 0$$

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We do this by splitting the log term as

$$\log|\cdots| = \frac{1}{2}\left(\log(\cdots)+\log\overline{(\cdots)}\right),$$

$+ k\pi i$

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and showing that the resulting two integrals are individually zero.

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Now, for X inside the unit disc, we have

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Need to show

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Thus, $2+\left(\frac{2(X^2-1)}{X+2}\right)$ is a nowhere vanishing holomorphic function in the unit disc $|X|\leq 1$,

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$$\oint \log\left(2+\left(\frac{2(X^2-1)}{X+2}\right)a\right)\frac{dX}{X+2}=0,$$

for all $|a|<1/2$.

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Thus, $2 + \left(\frac{2(X^2-1)}{X+2} \right)$ is a nowhere vanishing holomorphic function in the unit disc $|X| \leq 1$, which means that any branch of its complex log is holomorphic in the unit circle, and consequently,

$$\oint \log \left(2 + \left(\frac{2(X^2 - 1)}{X+2} \right) a \right) \frac{dX}{X+2} = 0,$$

for all $|a| < 1/2$.

As a function in the complex variable a, we have a holomorphic function that is identically zero on the open set $|a| < 1/2$, and we conclude that it is identically zero for all $a \in \mathbb{C}$

Need to show

$$\oint\oint\oint \log\left|\frac{2(X^2+X+1+(X^2-1)(y+z))}{X+2}\right|\left(\frac{dX}{X+2}\right)\frac{dy}{y}\frac{dz}{z}=0$$

Similarly, we now aim to show that

$$\oint \log\left(1+\left(\tfrac{1-X^2}{X(1+2X)}\right)\overline{a}\right)\tfrac{dX}{X+2}=0$$

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Again, for $|a|<1/2$, the log-term is holomorphic and $\frac{1}{2t(t+1/2)}$ has simple poles at $t=0$ and $t=-1/2$. Thus, by the residue theorem, we see that the required integral equals

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$$2\pi i\left(\log\left(1+\frac{0-1}{0+2}\,\overline{a}\right)\frac{1}{2(0+{}^1/_2)}+\log\left(1+\frac{{}^1/_4-1}{-{}^1/_2+2}\,\overline{a}\right)\frac{1}{2(-{}^1/_2)}\right)=0$$

The same arguments work for the other transformations

$$x+1+(x-1)(y+z) \xrightarrow{x=\frac{X(2X+1)}{X+2}} 2\,\frac{X^2+X+1+(X^2-1)(y+z)}{X+2}$$

$$x+1+(x-1)(y+z) \xrightarrow{x=\frac{X(2X^2-X+1)}{-(X^2-X+2)}} 2\,\frac{X^3-X^2+X-1+(X^3+1)(y+z)}{-(X^2-X+2)}$$

$$x+1+(x-1)(y+z) \xrightarrow{x=\frac{X(2X^3-X^2-X+1)}{-(X^3-X^2-X+2)}} 2\,\frac{X^4-X^3+X-1+(X^4-X^2+1)(y+z)}{-(X^3-X^2-X+2)}$$

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So the natural question is what transformations work, and what kind of polynomials do they result in?

Generalizations

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As before, we start with Condon's polynomial
$P(x, y, z) = x + 1 + (x - 1)(y + z)$ with Mahler measure $\frac{28}{5\pi^2}\zeta(3)$.

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$$x = \frac{f(X)}{g(X)}.$$

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This will lead us to an expression for the Mahler measure of the polynomial given by

$$f(X) + g(X) + (f(X) - g(X))(y + z),$$

in terms of the Mahler measure of Condon's polynomial P and the Mahler measure of $g(X)$.

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Notation

If $p(X) = \sum_{i=1}^{d} p_i X^i$, then define

$$\overline{p}(X) := \sum_{i=1}^{d} \overline{p_i} X^i.$$

In order for our methods to work, we need the following conditions on f and g:

1. If α is a root of $g(X)$, then $|\alpha| > 1$.
2. We require $f(X)$ to be of the form

$$f(X) = \pm X^k \overline{g}(X^{-1}),$$

where k is an integer that is at least equal to the degree of $g(X)$.

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Reasons for These Conditions

$$\oint\oint\oint_{|x|=1} \log|x+1+(x-1)(y+z)|\,\frac{dx}{x}\frac{dy}{y}\frac{dz}{z}$$

$$\Big\downarrow {\scriptstyle x=\frac{f(X)}{g(X)}}$$

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We need these conditions in order to make sure that the transformation preserves the unit circle.

Reasons for These Conditions

1 $g(x)$
all roots
outside
unit circle

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We need these conditions in order to make sure that the transformation preserves the unit circle. In particular, the map

$$\phi:\mathbb{S}^1\to\mathbb{S}^1$$
$$x\mapsto\frac{f(X)}{g(X)},$$

$f(x) = \pm x^k \bar{g}(x^{-1})$

k elements

$\phi^{-1}(\alpha) = \{ \quad \}$

is a <u>k-to-one</u> surjection from the unit circle onto itself.

Theorem (Lalín & N., 2021++)

Let $g(X) \in \mathbb{C}[X]$ *with all roots outside the unit disc, and let* $f(X) = \pm X^k \overline{g}(X^{-1})$, *for any integer* $k \geq \mathrm{degree}\, g$. *Then*

$$m(f + g + (f - g)(y + z)) = \frac{28}{5\pi^2}\zeta(3) + m(g).$$

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$$\implies m\left(X^2 + X + 1 + (X^2 - 1)(y + z)\right) = \frac{28}{5\pi^2}\zeta(3).$$

$m(g) = \log 2$

- $g(X) = X^3 - X^2 + X - 2$, we obtain

$$m(X^4 - X^3 + X^2 - X + 1 + (X^4 - 1)(y + z)) = \frac{28}{5\pi^2}\zeta(3).$$

Further Questions

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$x = \frac{f}{g}$ $x = x^m$

- We began with Condon's polynomial as the base case. Can the same method be used starting with other polynomials whose Mahler measure is known? $m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$

Further Questions

- We began with Condon's polynomial as the base case. Can the same method be used starting with other polynomials whose Mahler measure is known?
- Can these methods be used to calculate the Mahler measure of non-isosceles polynomials of the form

$$A(x) + B(x)y + C(x)z.$$

$A + B(y+z) \longrightarrow$

Identities involving Elliptic Curves

$P(x,y,z)$

$$m(x+1+(x-1)y+(x^2-1)z) \stackrel{?}{=} -\frac{3}{2}L'(E_{20a1},-1)$$

$$m((x+1)^2+(x^2-1)y+(x-1)^2z) \stackrel{?}{=} -\frac{2}{5}L'(E_{48a1},-1)$$

$$m(x^2-x+1+(x+1)y+(x+1)z) \stackrel{?}{=} -\frac{2}{9}L'(E_{30a1},-1)-\frac{476}{27}\zeta'(-2)$$

γ: $\{ P(x,y,z)=0 \quad |x|=1,\ |y|=1,\ |z|\geq 1 \}$

$S\gamma$: '' $\cdot\ |z|=1$

$$P(x,y,z)=0=P(x^{-1},y^{-1},z^{-1})$$

Eliminate z $\longrightarrow$ x,y $\longrightarrow$ corresponds to Elliptic curve

Maillot Curve

THANK YOU!